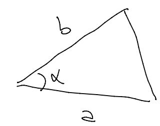
How do we measure the area of a region in plane?

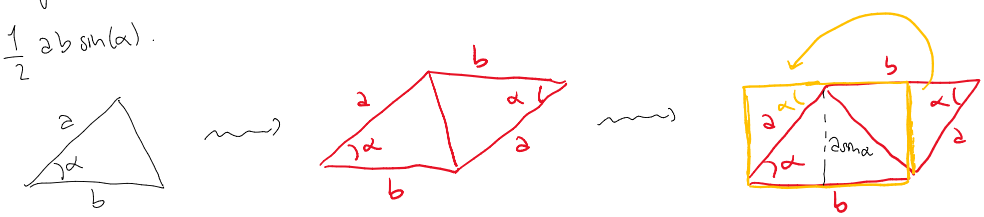
In order to measure areas of regions, we shall make the following assumptions regarding the notion of area:

- The area of a region is a non-negative real number.
- The area of a rectangle with perpendicular sides having lengths a and b is ab.
- If two regions can be obtained from each other by translations and rotations, then they have the same area.
- If we split a region into (possibly countably many) disjoint pieces, then the area of the region is the sum of the areas of the pieces

Fact: Using these assumptions, one can show that the area of the triangle



is $\frac{1}{2} ab \sin(\alpha)$.

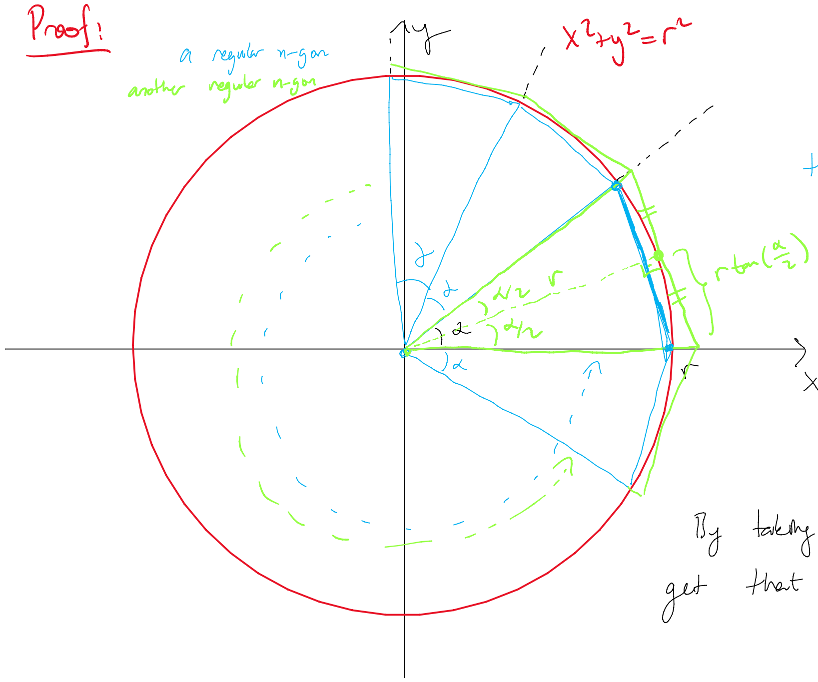


the area of the rectangle is $ab\sin(\alpha)$

After making these assumptions, we can find the areas of "well-behaved" regions by "approximating" these regions via "basic shapes" from outside and from within.

Example: let us prove that the area of a disc with radius r is πr^2.

Proof:



let $n \geq 3$ be an integer and let $\alpha = \frac{2\pi}{n}$

the area of the blue polygon $\leq$ the area of the disc with radius r $\leq$ the area of the green polygon

$$n \cdot \frac{1}{2} r^2 \sin(\alpha) \qquad\qquad n \cdot \frac{1}{2}\, 2r\tan\left(\frac{\alpha}{2}\right) r$$

By taking the limit of all sides as $n \to \infty$, we get that

$$\lim_{n\to\infty} n \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\overset{t}{\frac{2\pi}{n}}\right)}{\underset{t}{\frac{\pi 2}{n}}} \pi \cdot r^2 = \lim_{t\to 0} \underbrace{\cancel{\frac{\sin(t)}{t}}}_{\to 1} \cdot \pi r^2 = \pi r^2$$

$$\lim_{n\to\infty} n \frac{1}{\cancel{2}} \cancel{2}r \tan\left(\frac{\alpha}{2}\right) \cdot r = \lim_{n\to\infty} \underbrace{\cancel{\frac{\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}\pi}}}_{\to 1} \cdot \pi \cdot r^2 = \pi r^2$$

By the squeeze theorem, we get that the area of a disc with radius r is πr^2.